October Math Gems

Problem of the week 16

§1 Problems

Problem 1.1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

Problem 1.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xy(f(x+y) - f(x) - f(y)) = 2f(xy)$$

for all $x, y \in \mathbb{R}$.

Problem 1.3. In a triangle ABC with $sinA = cosB = cotC$. Find the value of $cosA$.

Problem 1.4.

$$cos^{-1}\frac{\sqrt{27+7\sqrt{5}}+\sqrt{27-7\sqrt{5}}}{14}$$

Problem 1.5. Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 = 3x - 12y + 50, \\ y^3 = 12y + 3z - 2, \\ z^3 = 27z + 27x. \end{cases}$$

Problem 1.6. Let a and b be real numbers such that $\sin a + \sin b = \dfrac{\sqrt{2}}{2}$ and $\cos a + \cos b = \dfrac{\sqrt{6}}{2}$. Find $\sin(a+b)$.

Problem 1.7. In triangle $ABC, AB = \sqrt{30}, AC = \sqrt{6}$, and $BC = \sqrt{15}$. There is a point D for which $\overline{AD}$ bisects $\overline{BC}$ and $\angle ADB$ is a right angle. The ratio

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)}$$

can be written in the form m/n, where m and n are relatively prime positive integers. Find $m+n$.

Problem 1.8. find area of triangle with sides $\sqrt{a^2+b^2}, \sqrt{c^2+a^2}, \sqrt{b^2+c^2}$, given a,b,c are positive. PS:- (don't use heron's formula, and $\frac{1}{2}$product of sides and product of sine of angle between these sides

Problem 1.9. Let ABC be a triangle such that $\angle B = 20^\circ$ and $\angle C = 40^\circ$. Also, length of bisection of angle A is equal to 2. Find the value of $BC - AB$?

Problem 1.10. Given acute $\triangle ABC$. Given point N such that$\angle NBA = \angle NCA = 90^\circ$. D and E are points on AC and AB, respectivelly, such that$\angle BNE = \angle CND$.Lines DE and BC intersects in F, and K is midpoint of segment DE. If X is point of intersection of circumcircles of $\triangle ABC$ and $\triangle ADE$, distinct from A, prove that $\angle KXF = 90^\circ$.

Problem 1.11. Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x+y+z &= 3, \\ x^2+y^2+z^2 &= 3, \\ x^3+y^3+z^3 &= 3. \end{aligned}$$

Problem 1.12. Find $ax^5 + by^5$ if the real numbers a, b, x, and y satisfy the equations

$$\begin{array}{rcl} ax+by &=& 3, \\ ax^2+by^2 &=& 7, \\ ax^3+by^3 &=& 16, \\ ax^4+by^4 &=& 42. \end{array}$$

Problem 1.13. Find all real numbers a, b, c, d such that

$$\begin{cases} a+b+c+d = 20, \\ ab+ac+ad+bc+bd+cd = 150. \end{cases}$$

Problem 1.14. Determine all real numbers $x > 0$ for which

$$\log_4 x - \log_x 16 = \frac{7}{6} - \log_x 8$$

Problem 1.15. Solve in the real numbers the equation $3^{\sqrt[3]{x-1}}\left(1 - \log_3^3 x\right) = 1$.

Problem 1.16. Evaluate the following sum

$$\frac{1}{\log_2 \frac{1}{7}} + \frac{1}{\log_3 \frac{1}{7}} + \frac{1}{\log_4 \frac{1}{7}} + \frac{1}{\log_5 \frac{1}{7}} + \frac{1}{\log_6 \frac{1}{7}} - \frac{1}{\log_7 \frac{1}{7}} - \frac{1}{\log_8 \frac{1}{7}} - \frac{1}{\log_9 \frac{1}{7}} - \frac{1}{\log_{10} \frac{1}{7}}$$

Problem 1.17. Solve in $\mathbb{R}$ the equation: $\log_{278}(\sqrt{x} + \sqrt[4]{x} + \sqrt[8]{x} + \sqrt[16]{x}) = log_2 \sqrt[16]{x}$.

Problem 1.18. Solve in real numbers: $log_{\sqrt{x}}(log_2(4^x - 2)) \leq 2$

Problem 1.19. Find all positive integers a, b such that

$$\frac{a^b + b^a}{a^a - b^b}$$

is an integer.

Problem 1.20. Find x if $\sqrt{x} = \sqrt[\sqrt{x}\sqrt{x}^5]{5}$
